

Nanopteron solution of the Korteweg-de Vries equation with an application in ion-acoustic waves

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Abstract: The nanopteron, which is a permanent but weakly nonlocal soliton, has been an interesting topic in numerical study for more than three decades. However, analytical solution of such a special soliton is rarely considered. In this paper, we study the explicit nanopteron solution of the Korteweg-de Vries (KdV) equation. Starting from the soliton-cnoidal wave solution of the KdV equation, the nanopteron structure is shown to exist. It is found that for suitable choice of wave parameters the soliton core of the soliton-cnoidal wave trends to be the classical soliton of the KdV equation and the surrounded cnoidal periodic wave appears as small amplitude sinusoidal variations on both side of the soliton core. Some interesting features of the wave propagation are revealed. In addition to the elastic interactions, it is surprising to find that the collision-induced phase shift of the cnoidal periodic wave is always half of its wavelength and this conclusion is universal to the soliton-cnoidal wave interaction. In the end, the nanopteron structure of the KdV equation is revealed in a plasma physics system. It is confirmed that the influence of plasma parameters on the nanopteron structure is in agreement with the classical soliton.

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1 Introduction

The Korteweg-de Vries (KdV) equation

$$u_t + Au u_x + Bu_{xxx} = 0, \quad (1)$$

which was originally derived to describe the propagation of gravity waves in shallow water [1], is now regarded as one of the most important systems in soliton theory. It arises as a fundamental model in diverse branches of physics, such as nonlinear optics, Bose-Einstein condensates and hydrodynamics [2].

In particular, the KdV equation plays a significant role in the study of small but finite amplitude ion acoustic waves, magnetoacoustic waves, Alfvén waves in plasma physics [3]. It has been reported in an experimental observation that dynamical properties of dust acoustic waves are found to agree quite well, particularly at low amplitudes and low Mach numbers, with the classical soliton solution of the KdV equation [4].

Since the dramatic discovery of the particlelike behavior of the localized waves by Zabusky and Kruskal in 1965 [5], there has been an unprecedented burst of research activities on solitons. Several effective methods, such as the inverse scattering transformation method [6], the Hirota's bilinear formalism [7], the Darboux transformation (DT) [8] and the Bäcklund transformation (BT) [9] have been developed to find the multiple soliton solutions of the KdV equation as well as other integrable systems. Besides multiple soliton solutions, interaction between solitons and other types of nonlinear waves is another topic of great interest [10–15]. Recently, by combining the symmetry reduction method with the DT or BT related nonlocal symmetries, researchers have established the interaction solutions between solitons and cnoidal periodic waves of the KdV equation [11] as well as the nonlinear Schrödinger equation [12]. Meanwhile, hinted by these results, two equivalent simple direct methods, the truncated Painlevé and the generalized tanh function expansion approach, are developed to find interaction solutions between solitons and other types of nonlinear waves, such as cnoidal waves, Painlevé waves, Airy waves and Bessel waves [13, 14].

In this paper, we report a special weakly nonlocal soliton solution called nanopterion of the KdV equation explicitly. The concept of nanopterion was originally introduced by Boyd, when studying a weakly nonlocal soliton in the ϕ^4 model numerically. It is a quasisoliton which almost satisfies the classical definition of a soliton, but fails because of small amplitude oscillatory tails extending to infinity in space [16]. During the past decades, the nanopterion structure in both continuous and discrete systems has been studied extensively [16–30]. For instance, Hunter and Scheurle have shown asymptotically that capillary-gravity water waves can be consistently modelled by a singularly perturbed Korteweg-de Vries equation and solutions of this wave equation are of nanopterion type when the Bond number is less than one third [19]. Actually, the investigation of interaction between a topological soliton and a background small amplitude wave has been an important topic in condensed matter physics for more than three decades [20]. In particular, it was shown that a small amplitude oscillatory wave can propagate transparently through a standing topological soliton with phase shift. In addition, the nanopterion structure has also been investigated in plasma physics [31, 32]. Keane et al. studied the Alfvén solitons in a fermionic quantum plasma numerically [32]. Starting from the governing equations for Hall magnetohydrodynamics including quantum corrections, coupled Zakharov-type equations are derived and numerically solved for time-independent and time-dependent cases. The time-independent Alfvén density soliton takes the form as an approximately Gaussian peak surrounded by smaller sinusoidal variations, which is just a nanopterion structure. Then taking the time-independent nanopterion solution as an initial condition, it was numerically confirmed that the shape of the Gaussian peak retains the same profile during its interaction with surrounded sinusoidal variations. Obviously, some of the above results suggest that the

interaction between a soliton and a small amplitude background wave is elastic, or else, the moving waves will degenerate during their propagation. However, analytical solution of the nanopterion structure and its exact relation to the classical soliton solution are rarely considered, which motivate a consideration of nanopterion solution of KdV equation in this paper.

In the next section of this paper, exact solution of the KdV equation describing a soliton dressed by a cnoidal periodic wave is obtained. The dressed structure and interaction between the soliton core and the background cnoidal periodic wave are also discussed in this section. In section 3, quasisoliton behavior of the soliton-cnoidal wave solution is considered. It is confirmed that, under certain limit condition, the soliton-cnoidal wave structure trends to be a nanopterion and there is an intimate connection between the nanopterion solution and the classical soliton solution. In section 4, the nanopterion solution of KdV equation are applied in a plasma physics system. The last section is a short summary and discussion.

2 Soliton-cnoidal wave solution of the KdV equation

Balancing the highest nonlinearity and dispersive terms in the KdV equation, we assume that the solution takes the following generalized truncated tanh expansion

$$u = u_0 + u_1 \tanh(w) + u_2 \tanh(w)^2, \quad (2)$$

where u_0 , u_1 , u_2 and w are functions of (x, t) to be determined later.

Substituting Eq. (2) into the KdV equation and collecting coefficients of different powers of $\tanh(w)$, we obtain the equation that u_0 , u_1 , u_2 and w need to satisfy. By requiring that the coefficients of the different powers of $\tanh(w)$ be separately equal to zero, we obtain six overdetermined equations for only four undetermined functions. It is fortunate to find that the last three of these over-determined equations are consistent. From the coefficients of $\tanh(w)^5$, $\tanh(w)^4$ and $\tanh(w)^3$, we find that u_2 , u_1 and u_0 can be solved as

$$u_2 = -\frac{12Bw_x^2}{A}, \quad (3)$$

$$u_1 = \frac{12Bw_{xx}}{A}, \quad (4)$$

$$u_0 = \frac{B}{A} \left(\frac{3w_{xx}^2}{w_x^2} - \frac{4w_{xxx}}{w_x} + 8w_x^2 \right) - \frac{w_t}{Aw_x}. \quad (5)$$

Consequently, the solution (2) can be reformed in terms of w

$$u = \frac{B}{A} \left[12(w_{xx} \tanh(w) + w_x^2 \tanh^2(w)) - \frac{w_t}{Bw_x} + \frac{4w_{xxx}}{w_x} - \frac{3w_{xx}^2}{w_x^2} - 8w_x^2 \right]. \quad (6)$$

Then, from the coefficient of $\tanh(w)^2$, we obtain the associated compatibility condition of w

$$\frac{w_t}{w_x} + B \left(\frac{w_{xxx}}{w_x} - \frac{3w_{xx}^2}{2w_x^2} - 2w_x^2 \right) + \lambda = 0, \quad (7)$$

with λ is an integral constant. Finally, one can verify that the two over-determined equations of powers $\tanh(w)^1$ and $\tanh(w)^0$ are identically satisfied by using Eqs. (3), (4), (5) and (7).

In order to find the interaction solution between a soliton and a cnoidal wave, we make the following ansatz of the solution in Eq. (7)

$$\begin{aligned} w &= \xi + c_1 \operatorname{arctanh}(c_2 S_n(\eta, m)), \\ \xi &= \frac{x - V_1 t}{W_1}, \quad \eta = \frac{x - V_2 t}{W_2}, \end{aligned} \quad (8)$$

where S_n is the usual Jacobi elliptic sine function and the parameter m is known as its modulus. V_1 and V_2 are velocities of the soliton and its surrounded cnoidal wave, respectively. W_1 and W_2 are quantities related to soliton width and cnoidal wavelength, respectively.

Substituting the ansatz (8) back into Eqs. (6) and (7), the following two results are revealed as expected. First, we find that if we take the elliptic modulus m and velocities V_1, V_2 as arbitrary, the other wave parameters $\{W_1, W_2, c_1, c_2, \lambda\}$ can be determined by vanishing the coefficients of the different powers of Jacobi elliptic functions

$$\begin{aligned} W_1 &= \sqrt{\frac{8B(1-m^2)}{V_1 - V_2}}, \quad W_2 = \sqrt{\frac{2B(1-m^2)}{V_1 - V_2}}, \\ \lambda &= \frac{(m^2 - 5)V_1 + (3m^2 + 1)V_2}{4(m^2 - 1)}, \\ c_1 &= \frac{\delta}{2}, \quad c_2 = m, \quad \delta^2 = 1. \end{aligned} \quad (9)$$

Actually, the wave parameters $\{W_1, W_2, c_1, c_2, \lambda\}$ may have five branches of solutions. However, we would go no further to the other four branches of parameter selections because singularities of u occur for any of them. Second, the explicit soliton-cnoidal wave solution of the KdV equation is obtained by combining Eqs. (6), (8) and (9),

$$\begin{aligned} u &= \frac{3(V_1 - V_2)}{2AG^2} \left[\frac{(m^2 - 1 + 2G)^2}{(m^2 - 1)} \tanh(w)^2 - 2\delta m S(m^2 - 1 + 2G) \tanh(w) + m^2 - 1 \right] \\ &\quad - \frac{(m^2 + 7)V_1 - (3m^2 + 5)V_2}{2A(m^2 - 1)}, \end{aligned} \quad (10)$$

with

$$G = 1 - m^2 S_n^2 + \delta m C_n D_n, \quad S \equiv S_n(\eta, m), \quad C_n \equiv C_n(\eta, m), \quad D_n \equiv D_n(\eta, m).$$

From the soliton-cnoidal wave solution (10), it is obvious that the range of modulus m should be $0 \leq m < 1$. The exclusion of $m = 1$ means that the solution (10) does not have a two-soliton limit. However, this exclusion may introduce some new features. For instance, the head-on collision between a soliton and a cnoidal periodic wave can also be modeled by the solution (10), while only overtaking collision can occur for the multiple soliton solution of the KdV equation.

As pointed out in our previous paper [12], a soliton-cnoidal wave can be viewed as a dressed soliton, that is, a soliton dressed by a cnoidal periodic wave. Consequently, the soliton-cnoidal wave can be

divided into two parts. By taking the limit $\tanh(w) = \pm 1$ in Eq. (10), we obtain the cnoidal periodic wave part of the dressed soliton

$$C_L = \frac{3(V_1 - V_2)(1 + \delta m S_n)(m^2 - 1 + 2G)}{AG^2} + \frac{(5 - m^2)V_1 + (3m^2 - 7)V_2}{2A(m^2 - 1)},$$

$$x - V_1 t < 0, \quad (11)$$

and

$$C_R = \frac{3(V_1 - V_2)(1 - \delta m S_n)(m^2 - 1 + 2G)}{AG^2} + \frac{(5 - m^2)V_1 + (3m^2 - 7)V_2}{2A(m^2 - 1)}.$$

$$x - V_1 t > 0. \quad (12)$$

Correspondingly, the soliton part of the wave is

$$S_L = u - C_L = -\frac{3(V_1 - V_2)(m^2 - 1 + 2G)}{2AG^2} \left[\frac{(m^2 - 1 + 2G)}{(m^2 - 1)} \text{sech}(w)^2 + 2\delta m S(\tanh(w) + 1) \right],$$

$$x - V_1 t < 0, \quad (13)$$

and

$$S_R = u - C_R = -\frac{3(V_1 - V_2)(m^2 - 1 + 2G)}{2AG^2} \left[\frac{(m^2 - 1 + 2G)}{(m^2 - 1)} \text{sech}(w)^2 + 2\delta m S(\tanh(w) - 1) \right],$$

$$x - V_1 t > 0. \quad (14)$$

To illustrate this point more clearly, let us see some figures. Fig. 1(a) exhibits the soliton-cnoidal wave structure of u determined by Eq. (10) at $t = 0$. Fig. 1(b) and Fig. 1(c) reveal the related structures of the cnoidal periodic wave and the soliton core of u , respectively. Obviously, the superposition of Fig. 1(b) and Fig. 1(c) is just Fig. 1(a). It is observed from Fig. 1(b) that apart from the soliton center, the solution rapidly tends to a cnoidal periodic wave. It is clear from Fig. 1(c) that after removing the periodic wave background from u , the left is just a soliton structure given by Eqs. (13) and (14). Fig. 1(d) shows an elastic overtaking collision process between a soliton and a cnoidal wave where both of them are right-going and the soliton is traveling faster. It can be concluded from Fig. 1(d) that despite the cnoidal periodic wave is a delocalized structure, in the space-time evolution of the soliton-cnoidal wave, every peak of the cnoidal periodic wave elastically interacts with the soliton core except for a phase shift. For the plot of Fig. 1, the selection of nonlinearity coefficient A and dispersion coefficient B are given by

$$A = \frac{48619\sqrt{1973}\sqrt{1980}}{315778650} \simeq 0.304, \quad B = \frac{388110739\sqrt{1973}\sqrt{1980}}{464096952000} \simeq 1.653, \quad (15)$$

which is derived from the KdV equation describing the propagation of ion acoustic waves in section 4.

The dressed structure enables us to compute the collision-induced phase shift of the cnoidal periodic wave. If we consider an overtaking collision process between a right-going soliton and a right-going cnoidal periodic wave where the soliton is traveling faster as depicted in Fig. 1. At time $t = 0$, the cnoidal periodic wave peaks on left side of soliton core have interacted with the soliton core, while the cnoidal periodic peaks on right side of soliton core are not. So there is a phase shift between them.

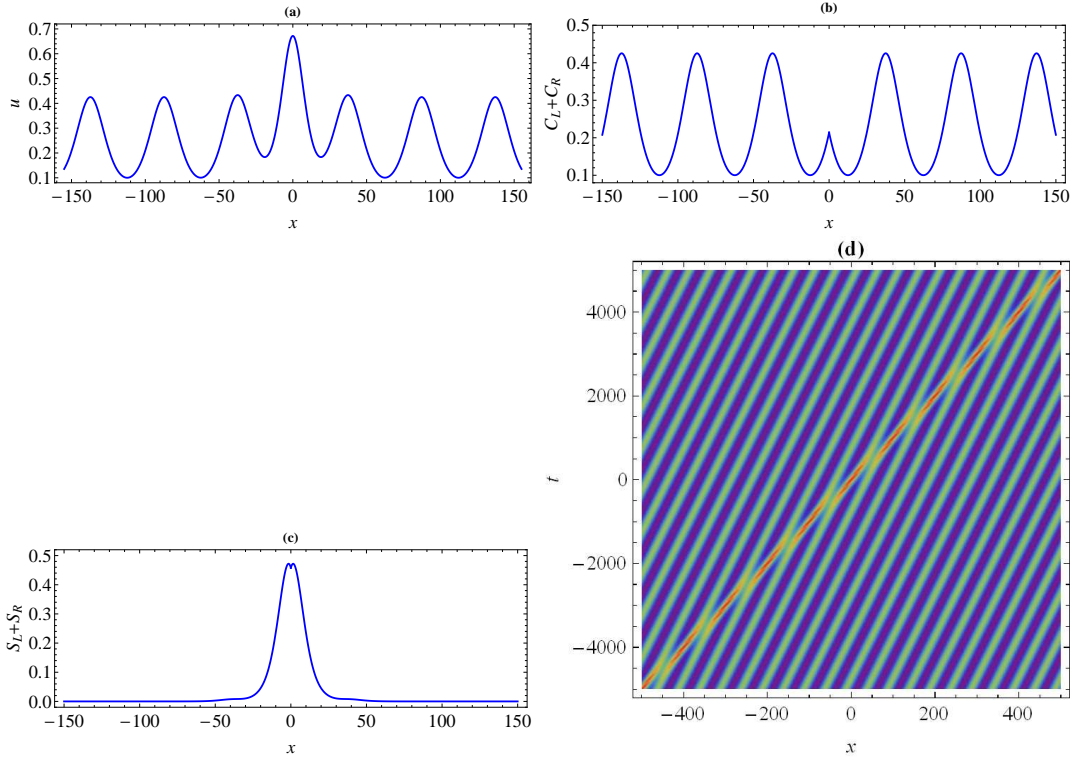


FIG.1 (a) The stationary soliton-cnoidal wave structure of the KdV equation given by Eq. (10). (b) The related cnoidal periodic wave structure given by Eqs. (11) and (12). (c) The related soliton structure given by Eqs. (13) and (14). (d) The density of u for space-time evolution. The parameters are $m = 0.3$, $V_1 = 0.1$, $V_2 = 0.05$, and $\delta = 1$. The coefficients A and B of the KdV equation are given by (15).

Obviously, the cnoidal periodic wave peaks on left and right side of soliton core can be expressed by C_L and C_R , respectively. By using Eqs. (11) and (12), it is surprising to find that the collision-induced phase shift of the cnoidal periodic wave is given by

$$\Delta_{cnoidal} = 2W_2K(m) = \frac{\lambda_c}{2}, \quad (16)$$

where λ_c is the wavelength of the cnoidal periodic wave, $K(m)$ is the first kind of complete elliptic integral. This result can be easily verified. By substituting $\eta = \eta + 2K(m)$ in (11), we obtain

$$C'_L(\eta + 2K(m)) = \frac{3(V_1 - V_2)(1 + \delta m S'_n)(m^2 - 1 + 2G')}{AG'^2} + \frac{(5 - m^2)V_1 + (3m^2 - 7)V_2}{2A(m^2 - 1)}, \quad (17)$$

with $S'_n = S_n(\eta + 2K(m), m)$, $C'_n = C_n(\eta + 2K(m), m)$, $D'_n = D_n(\eta + 2K(m), m)$ and $G' = 1 - m^2 S_n'^2 + \delta m C'_n D'_n$. Expanding (17) and using the Jacobi elliptic identities

$$Sn(2K(m), m) = 0, \quad Cn(2K(m), m) = -1, \quad \text{and} \quad Dn(2K(m), m) = 1, \quad (18)$$

we can directly demonstrate that $C'_L(\eta + 2K(m)) = C_R(\eta)$. Incidentally, the period of the S_n and C_n functions is $4K(m)$, while the period of the D_n function is $2K(m)$. So that functions C_L and C_R are of period $4K(m)$. From formula (16), the phase shift in Fig. 1(b) can be determined as 24.95, which is in accordance with the figure.

The phase shift formula of (16) tells us that the phase shift of cnoidal periodic wave after its interaction with a soliton is always half of its wavelength. Significantly, this phase shift formula is universal to all the soliton-cnoidal wave solutions for different models in Refs. [11–13]. Unfortunately, it is still difficult to calculate phase shift of soliton because of the mixture of tanh and Jacobi elliptic functions. The existence of Jacobi elliptic functions prevent us from calculating the difference of the phase at the limits of time approaching negative and positive infinities. Therefore, an alternative method should be introduced to calculate the phase shift of the soliton.

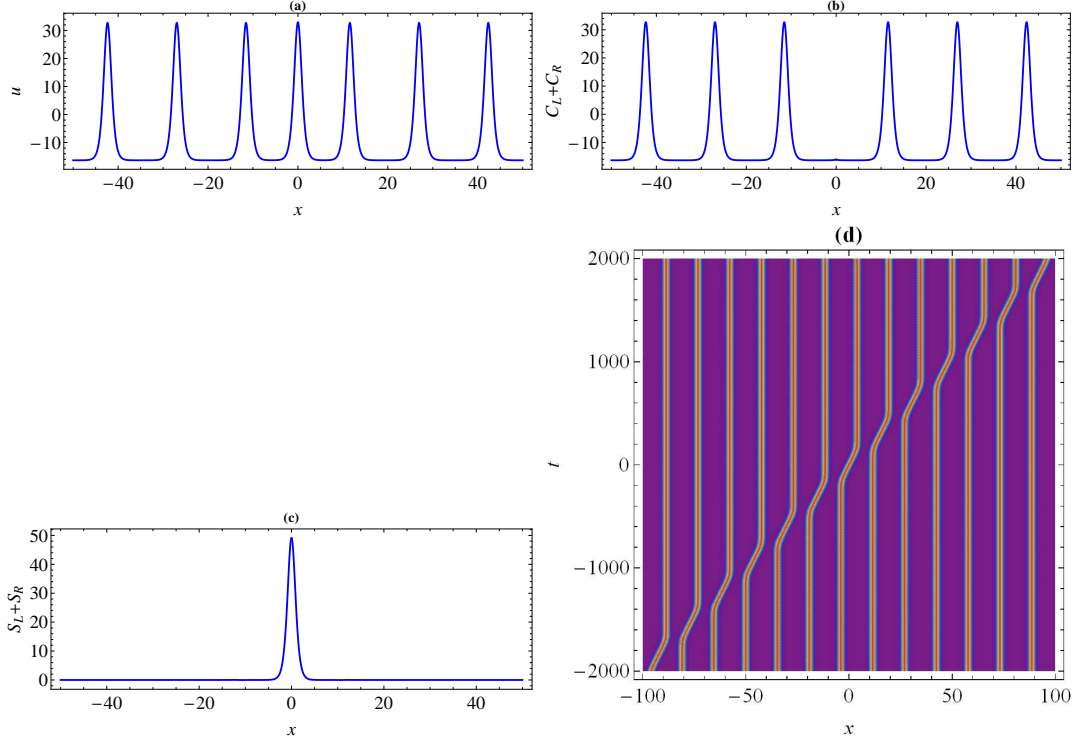


FIG.2 (a) The soliton-cnoidal wave structure with $m = 0.99$ at $t = 0$. (b) The related cnoidal periodic wave structure. (c) The related soliton structure. (d) The density of u for space-time evolution. The parameters are $m = 0.99$, $V_1 = 0.05$, $V_2 = 0$, and $\delta = 1$. The coefficients A and B of the KdV equation are given by (15).

Due to the fact that the parameter m appears as not only the modulus of the Jacobi elliptic functions but also its coefficient ($c_2 = m$), the amplitude of cnoidal periodic wave trends to thrive with increasing m . Under the limit $m \rightarrow 1$ ($m \neq 1$), the amplitude of the cnoidal periodic wave is of the same order of the soliton core. In Fig. 2(a), which is shown to illustrate this phenomenon at $t = 0$, it can be observed that as soon as the parameter m approaches 0.99, the amplitude of cnoidal wave becomes comparable to the soliton core. Fig. 2(b) shows that the solution (10) exponentially approaches the cnoidal waves under limit $x \rightarrow \pm\infty$. We also notice from Fig. 2(c) that, after the periodic wave peaks C_L and C_R are taken away from the exact solution u , only a tall and slim soliton structure is revived. Fig. 2(d) reveals that the soliton core and every peak of the standing cnoidal periodic wave can pass through each other transparently with phase shift.

3 Quasisoliton behavior as nanopterion

Before we proceeding further, let us first review the classical soliton solution of the KdV equation. By using the usual tanh expansion method, the single soliton solution of the KdV equation can be obtained as

$$u = \frac{1}{AW^2} [8B + VW^2 - 12B \tanh^2(\xi)], \quad \xi = \frac{x - Vt}{W}. \quad (19)$$

Imposing the boundary conditions

$$u \rightarrow 0, \quad u_\xi \rightarrow 0, \quad u_{\xi\xi} \rightarrow 0, \quad \text{as } \xi \rightarrow \pm\infty, \quad (20)$$

the width of soliton can be determined

$$u = \frac{3V}{A} \text{sech}^2 \left(\frac{x - Vt}{W} \right), \quad W = \sqrt{\frac{4B}{V}}. \quad (21)$$

Obviously, there is an intimate connection between the usual tanh function expansion method and the generalized tanh function expansion method. If we take w as a straight line solution, namely, $w = (x - Vt)/W$, the solution (6) reduces to the single soliton solution (19) obtained by the usual tanh function expansion method.

Now, let us consider the asymptotic behavior of soliton-cnoidal wave solution (10). Under the ultra limit condition $m = 0$, $v_1 = V$ and $v_2 = -V$, the wave parameters (9) degenerate to

$$W_1 = \sqrt{\frac{4B}{V}}, \quad W_2 = \sqrt{\frac{B}{V}}, \quad \lambda = \frac{3V}{2}, \quad c_1 = \frac{\delta}{2}, \quad c_2 = 0, \quad (22)$$

and the soliton-cnoidal wave solution (10) reduce to the classical soliton solution (21). From Eq. (22), it is interesting to notice that the substitution of straight line solution $w = (x - Vt)/W$ into the compatibility condition (7), the width of soliton can also be determined as $W = \sqrt{4B/V}$ with $\lambda = 3V/2$. The deeper physical reason why the compatibility condition of w plays the similar role as the boundary condition (20) is still need further consideration. This interesting limit case hints us to consider the asymptotic behavior of the soliton-cnoidal wave solution (10). Under asymptotic condition $v_1 = V$, $v_2 = -V$, and $m \rightarrow 0$, we find that the soliton core profile trends to be the classical soliton solution of the KdV equation and the surrounded conidal periodic wave appears as a small amplitude sinusoidal wave oscillating around zero. This very expression is just the nanopterion structure proposed by Boyd. However, in Boyd's definition, the surrounded small amplitude sinusoidal wave is a standing wave. To satisfy Boyd's definition, we only need to take the limit condition as $v_1 = 2V$ and $v_2 = 0$ $m \rightarrow 0$. In this limit condition, we find that the tails oscillating around V/A . Using the straightforward transformation

$$u' = u - \frac{V}{A},$$

we obtain the solution u' which satisfies a more classical version of the KdV equation

$$u'_t + (Au' + V)u'_x + Bu'_{xxx} = 0.$$

Actually, the nanopterion structure has been found with the soliton core and the background wave are traveling at the same velocity in a generalized version of the continuous Sine-Gordon equation [23] as well as in the nonlinear Klein-Gordon lattice [30]. In this paper, we find that for the nanopterion structure of the KdV equation the soliton core and the background wave are traveling at the same speed but in opposite directions. Remarkably, not every soliton-cnoidal wave solution of the KdV equation has the nanopterion limit. We call the limited soliton-cnoidal wave solution which behaves like a nanopterion as nanopterion solution.

Up to now, we can write down the nanopterion solution of the KdV equation explicitly. By taking $v_1 = V$, $v_2 = -V$, and $m \rightarrow 0$ in Eqs. (6), (9) and (10), we obtain

$$u = \frac{3V}{2AG^2} \left[\frac{(m^2 - 1 + 2G)^2}{(m^2 - 1)} \tanh(w)^2 - 2\delta m S(m^2 - 1 + 2G) \tanh(w) + m^2 - 1 \right] - \frac{2V(m^2 + 3)}{A(m^2 - 1)}, \quad m \rightarrow 0, \quad (23)$$

with

$$w = \frac{x - Vt}{W_1} + \frac{\delta}{2} \operatorname{arctanh}(m S_n(\frac{x + Vt}{W_2}, m)),$$

and the determined wave parameters are

$$\lambda = \frac{(3 - m^2)V}{2(1 - m^2)}, \quad W_1 = \sqrt{\frac{4B(1 - m^2)}{V}}, \quad W_2 = \sqrt{\frac{B(1 - m^2)}{V}}.$$

For the solution (23), the modulus m appears as an important parameter which affects the wave profile dominantly. Taking it approaches zero asymptotically, we obtain a special weakly nonlocal soliton as nanopterion with the oscillating tails can be minimized at $x \rightarrow \pm\infty$ and the soliton core gets more and more closer to the classical soliton. While when it approaches one asymptotically, the amplitude of the background wave becomes comparable to the soliton core which deviates from the the classical soliton solution largely. Thus, the only condition for the realization of nanopterion structure in solution (23) is the parameter m should be sufficiently small.

Under the limit $m \rightarrow 0$, the function $K(m)$ trends to $\pi/2$. Thus, the collision-induced phase shift of the small amplitude background wave can be approximately taken as

$$\Delta_{\text{sinusoidal}} = 2W_2 K(m) \simeq W_2 \pi. \quad (24)$$

The nanopterion solution also has a dressed structure. By taking the limit $\tanh(w) = \pm 1$ in Eq. (23), the small amplitude oscillating tails on the left and right side of the soliton core are

$$C_L = \frac{6V(1 + \delta m S_n)(m^2 - 1 + 2G)}{AG^2} + \frac{2(3 - m^2)V}{A(m^2 - 1)}, \quad x - Vt < 0, \quad (25)$$

and

$$C_R = \frac{6V(1 - \delta m S_n)(m^2 - 1 + 2G)}{AG^2} + \frac{2(3 - m^2)V}{A(m^2 - 1)}, \quad x - Vt > 0. \quad (26)$$

Correspondingly, the soliton core can be expressed as

$$S_L = u - C_L = -\frac{3V(m^2 - 1 + 2G)}{AG^2} \left[\frac{(m^2 - 1 + 2G)}{(m^2 - 1)} \text{sech}(w)^2 + 2\delta m S(\tanh(w) + 1) \right],$$

$$x - Vt < 0, \quad (27)$$

and

$$S_R = u - C_R = -\frac{3V(m^2 - 1 + 2G)}{AG^2} \left[\frac{(m^2 - 1 + 2G)}{(m^2 - 1)} \text{sech}(w)^2 + 2\delta m S(\tanh(w) - 1) \right].$$

$$x - Vt > 0. \quad (28)$$

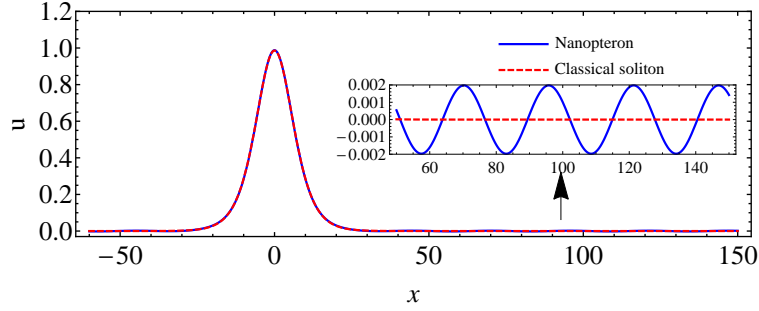


FIG.3 A comparison of the classical soliton solution (21) to the nanopteron solution (23). The wave parameters are

$V = 0.1$, $m = 0.001$, and $\delta = 1$. The coefficients A and B of the KdV equation are given by (15).

Fig. 3 presents a comparison of the classical soliton solution (21) to the nanopteron solution (23) with $m = 0.001$ at $t = 0$. As can be seen, the curves of them coincide exactly with each other at a large space-scale. However, the inset on the right side of Fig. 3 shows that the oscillating tails is nonvanishing despite of the tiny amplitude. When we takes m a little larger, the nanopteron tail shows up conspicuously. Fig. 4(a) presents a comparison of classical soliton to the nanopteron structures with $m = 0.02$ and $\delta = \pm 1$ at $t = 0$. With $\delta = 1$, the soliton core of the nanopteron is higher and slightly narrower than the classical soliton. While with the parameter $\delta = -1$, the soliton core of the nanopteron is shorter and slightly wider than the classical soliton. Interestingly, the crests and troughs of the surrounded sinusoidal waves of the nanopteron structures with $\delta = \pm 1$ are corresponding to each other. Fig. 4(b) and Fig. 4(c) reveal the dressed structure of nanopteron solution with $\delta = 1$. Fig. 4(b) shows that, apart from the soliton center, the solution rapidly approaches a small amplitude sinusoidal wave oscillating around zero. From Eq. (24), the phase shift of the small amplitude sinusoidal wave can be approximately determined as $W_2\pi \simeq 12.77$, which is in accordance with the Fig. 4(b). Figure 4(c) reveals that only soliton $S_L + S_R$ is left after the sinusoidal wave are ruled out from the exact solution u . Fig 4(d) is a three-dimensional plot of the nanopteron solution with $\delta = 1$.

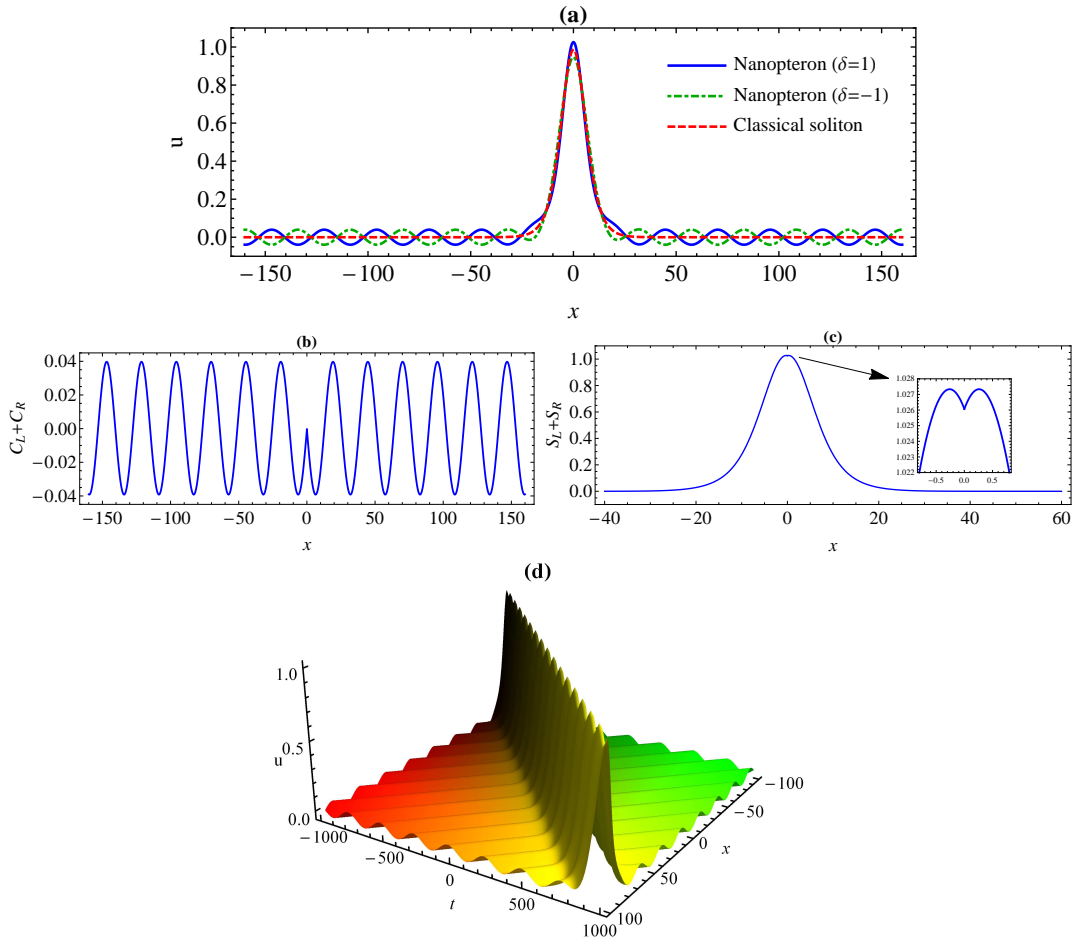


FIG.4 (a) A comparison of the classical soliton to the nanopteron structure with $\delta = \pm 1$. (b) The related small amplitude background wave with $\delta = 1$. (c) The related soliton structure with $\delta = 1$. (c) Three-dimension plot of the nanopteron solution (23) with $\delta = 1$ for space-time evolution. The other parameters are $m = 0.02$ and $V = 0.1$. The coefficients A and B are given by (15).

4 Application to ion acoustic waves

4.1 Basic equations and derivation of the KdV equation

In this section, we are modeling ion-acoustic (IA) waves in an ideal homogeneous magnetized plasma whose constituents are inertialess hot electrons and singly charged warm ions. The plasma is unbounded and immersed in a static magnetic field $\mathbf{B} = B_0 \hat{z}$, where \hat{z} is the unit vector along the \hat{z} -direction and B_0 is the strength of the magnetic field. At equilibrium, the charge neutrality implies $n_{e0} = n_{i0} = n_0$, where n_{e0} and n_{i0} are the undisturbed electron and ion number densities, respectively. In addition, we assume that the inertialess hot electrons have a kappa distribution

$$n_e = n_0 \left[1 - \frac{e\phi}{(\kappa - \frac{3}{2})T_e} \right]^{-\kappa + \frac{1}{2}},$$

where the real parameter κ ($\kappa > \frac{3}{2}$) measures the deviation from the Maxwell-Boltzmann equilibrium.

Under these conditions, the dynamics of the warm ion component are governed by the following normalized equations for adiabatic process

$$\frac{\partial n_i}{\partial t} + \nabla \cdot (n_i \mathbf{u}_i) = 0, \quad (29)$$

$$\frac{\partial \mathbf{u}_i}{\partial t} + (\mathbf{u}_i \cdot \nabla) \mathbf{u}_i = -\nabla \phi + \Omega \mathbf{u}_i \times \hat{z} - \gamma \sigma n_i^{-\frac{1}{3}} \nabla n_i, \quad (30)$$

$$\nabla^2 \phi = \left(1 - \frac{\phi}{\kappa - \frac{3}{2}}\right)^{-\kappa + \frac{1}{2}} - n_i, \quad (31)$$

where n_i is the warm ion number density normalized by n_0 , \mathbf{u}_i is the ion hydrodynamic velocity normalized by the IA wave speed $C_s = (T_e/m_i)^{1/2}$, and ϕ is the electrostatic potential normalized by T_e/e . Space and time variables are normalized by the Debye length $\lambda_{De} = (T_e/4\pi n_{e0}e^2)^{1/2}$, and the inverse ion plasma period $\omega_{pi}^{-1} = (m_i/4\pi n_{e0}e^2)^{1/2}$, respectively. Moreover, $\Omega = B_0/\sqrt{4\pi n_{e0}m_i}$ is the absolute value of the ion cyclotron frequency normalized by the ion plasma period ω_{pi} , $\sigma = T_i/T_e$ is the ratio of ion to electron temperature. The adiabatic index γ in the momentum equation equals 5/3, we keep it in symbol form for simplicity.

In the weak perturbation limit $\frac{e\phi}{T_e} \ll 1$, the Posson's equation (31) has the following approximation

$$\nabla^2 \phi \simeq (1 - n_i) + a_\kappa \phi + b_\kappa \phi^2, \quad \text{with} \quad a_\kappa = \frac{2\kappa - 1}{2\kappa - 3}, \quad b_\kappa = \frac{4\kappa^2 - 1}{2(2\kappa - 3)^2}.$$

We now employ the reductive perturbation technique to study the nonlinear dynamics of a small but finite amplitude IA waves, which requires the introducing of the following stretched coordinates

$$\xi = \epsilon^{1/2}(l_x x + l_y y + l_z z - V_p t), \quad \tau = \epsilon^{3/2} t, \quad (32)$$

where ϵ is a small real parameter measuring the weakness of the dispersion, V_p is an unknown wave phase speed to be determined later. The parameters l_x , l_y and l_z , which satisfy $l_x^2 + l_y^2 + l_z^2 = 1$, are the directional cosines of the wave vector \mathbf{k} along the x , y and z axes, respectively. Due to the anisotropy of the magnetic field, the perpendicular components of the IA velocity are expected to appear at lower order in ϵ than the parallel component. Thus, we expand all the perturbed quantities about their equilibrium values in the powers of ϵ as

$$n_i = 1 + \epsilon n_1 + \epsilon^2 n_2 + \dots, \quad (33)$$

$$\phi = \epsilon \phi_1 + \epsilon^2 \phi_2 + \dots, \quad (34)$$

$$u_{ix} = \epsilon^{\frac{3}{2}} u_{1x} + \epsilon^2 u_{2x} + \dots. \quad (35)$$

$$u_{iy} = \epsilon^{\frac{3}{2}} u_{1y} + \epsilon^2 u_{2y} + \dots. \quad (36)$$

$$u_{iz} = \epsilon u_{1z} + \epsilon^2 u_{2z} + \dots. \quad (37)$$

Substituting Eqs. (33)-(37) into the normalized set of fluid equations (29)-(31) and equating coefficients of ϵ to zero, we get compatibility conditions in the lowest order, from which n_1 , u_{1x} , u_{1y} , and u_{1z} , can be solved in terms of ϕ_1

$$n_1 = a_\kappa \phi_1, \quad (38)$$

$$\Omega u_{x1} = -(1 + \gamma \sigma a_\kappa) l_y \frac{\partial \phi_1}{\partial \xi}, \quad (39)$$

$$\Omega u_{y1} = (1 + \gamma \sigma a_\kappa) l_x \frac{\partial \phi_1}{\partial \xi}, \quad (40)$$

$$V u_{z1} = \frac{V^2 n_1}{l_z} = (1 + \gamma \sigma a_\kappa) l_z \phi_1. \quad (41)$$

From Eqs. (38) and (41), the unknown phase speed V_p can be determined as

$$V_p = l_z \left(\gamma \sigma + \frac{1}{a_\kappa} \right)^{\frac{1}{2}}. \quad (42)$$

It is evident from Eq. (42) that the increasing of ion to electron temperature ratio σ causes higher phase speed, while the increasing of obliqueness (decreasing l_z) causes lower phase speed. One can also readily prove that the decreasing of the population of the superthermal electrons (larger a_κ , $a_\kappa < 1$) make IA wave propagate with a smaller phase speed. Under limit case $\kappa \rightarrow \infty$ and $a_\kappa \rightarrow 1$, the Maxwellian behavior is recovered.

Combining the higher order terms and applying the lowest order solutions (38)–(42), We obtain the KdV equation

$$\frac{\partial \phi_1}{\partial \tau} + A \phi_1 \frac{\partial \phi_1}{\partial \xi} + B \frac{\partial^3 \phi_1}{\partial \xi^3} = 0, \quad (43)$$

with the nonlinearity coefficient A and dispersion coefficient B are

$$A = \frac{V_p}{1 + \gamma \sigma a_\kappa} \left(\frac{3a_\kappa^2 - 2b_\kappa}{2a_\kappa} - \frac{4\gamma \sigma a_\kappa^2}{3} \right), \quad (44)$$

$$B = \frac{V_p}{2} \left[\frac{(1 - l_z^2)(1 + \gamma \sigma a_\kappa)}{a_\kappa \Omega^2} + \frac{1}{(1 + \gamma \sigma a_\kappa) a_\kappa} \right]. \quad (45)$$

To study the IA nanopterion structure of the KdV equation (43), we only need to change coordinates from (x, t, u) to (ξ, τ, ϕ_1) in solution (23).

4.2 Parametric investigation

In this section, we numerically analyze the influence of the plasma parameters on the nanopterion profile. It is noticed that the soliton core as well as the surrounded sinusoidal wave peaks are significantly influenced by electron superthermality (via κ), ion to electron temperature ratio σ , magnetic field strength Ω and the direction cosine l_z . We also compare our results with earlier findings in which a classical soliton solution are considered.

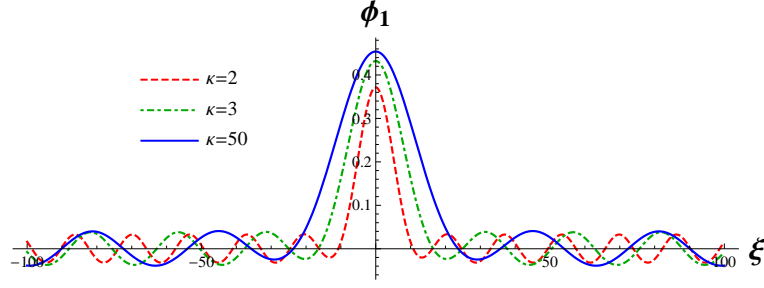


FIG.5 Variation of the IA nanopteron structure with respect to the electron superthermality κ . The wave parameters are $m = 0.04$, $\delta = -1$, and $V = 0.05$. The other plasma parameters are $\Omega = 0.3$, $\sigma = 0.01$ and $l_z = 0.3$.

Let us first discuss how nanopteron structures change with electron superthermality. Fig. 5 suggests that the amplitude and the width of the soliton core decrease as we introduce a higher proportion of kappa-distributed electrons (i.e., for decreasing κ). Similar effect on the surrounded small amplitude sinusoidal wave can also be observed in Fig. 5. This behavior happens due to the fact that the increase of electron kappa parameter causes the coefficient of the nonlinear coefficient A to decline, as a result, the amplitude of the nanopteron go up. On the other hand, the increase of the electron superthermal parameter κ also causes the dispersion coefficient B to be larger, as a result, the width of the nanopteron become wider. Under the Maxwellian limit ($\kappa \rightarrow \infty$), the nanopteron structures approach a constant value. Similar results of the electron superthermallity on the structure of a classical soliton are observed in Refs. [33]- [35].

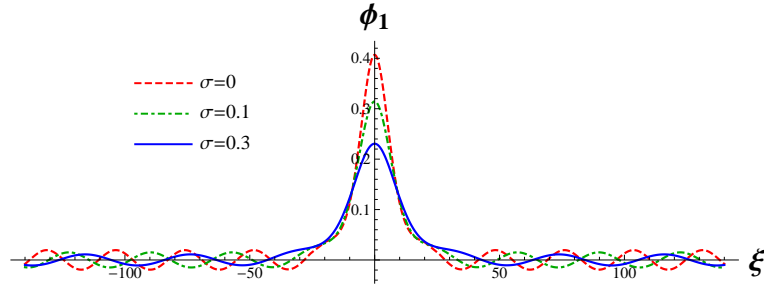


FIG.6 Variation of the IA nanopteron structure with respect to the ion to electron temperature ratio σ for $m = 0.025$, $\delta = 1$, $V = 0.04$, $\Omega = 0.3$, $\sigma = 0.01$, $l_z = 0.3$, and $\kappa = 3$.

In Fig. 6, variation of the nanopteron profile with the ion to electron temperature ratio σ is shown. It is evident from the figure that the amplitude of the soliton core decrease as the ion temperature thrives, while width of them trends to be larger. It can also be observed that the variations of the surrounded oscillating tails are in accordance with the soliton core. This result is in agreement with Ref [36].

Fig. 7 shows how the nanopteron shape changes with varying magnetic field strength Ω . Fig. 7(a) shows a range of values of Ω smaller than 1, and Fig.7(b) shows values of Ω range from 1 to 10. As can be seen in Fig. 7(a), the soliton core and the surrounded sinusoidal wave peaks become narrower as Ω increases. Then in Fig. 7(b), as Ω increases further in value, the curves trends to converge, suggesting

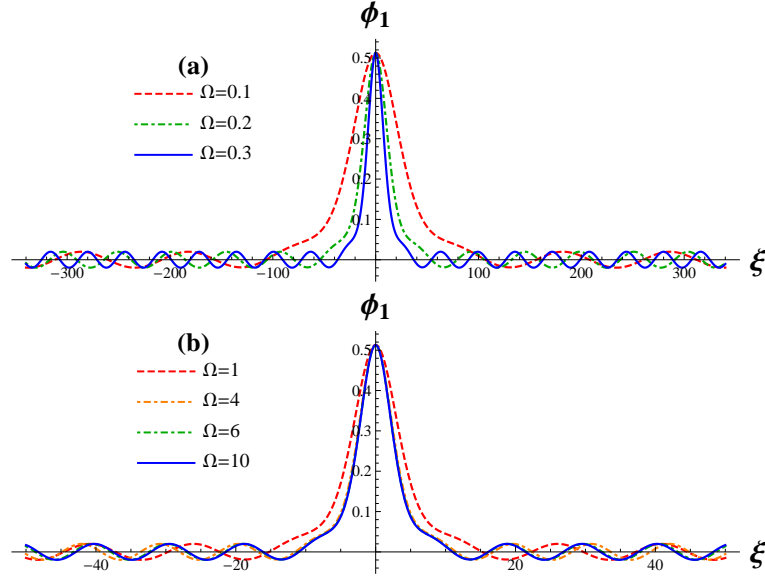


FIG.7 Variation of the nanopteron with respect to the electron magnetic field strength Ω for $m = 0.02$, $\delta = 1$, $V_1 = 0.05$, $V_2 = -0.05$, $\Omega = 0.3$, $\sigma = 0.01$, $l_z = 0.3$, $\kappa = 50$.

that the nanopteron shape becomes constant. This behavior can be verified from expression of dispersion coefficient B . Under the limit $(1 - l_z^2)(\gamma\sigma + 1/a_\kappa) \ll \Omega^2$, B can be approximately taken as

$$B \simeq \frac{V_p}{2} \left[\frac{1}{(1 + \gamma\sigma a_\kappa)a_\kappa} \right], \quad (46)$$

which is not affected by the magnetic field strength Ω . Obviously, the amplitude of the nanopteron keeps constant due to the Ω -independence of nonlinear coefficient A . Similar effects of the magnetic field strength Ω on a classical soliton are observed in Refs. [35]- [37].

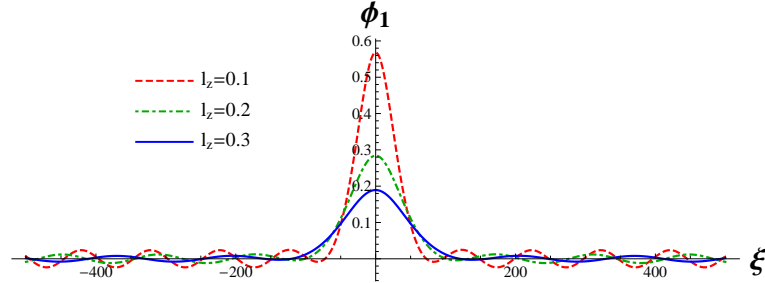


FIG.8 Variation of the soliton-cnoidal wave for the slow IA mode with respect to the electron superthermality κ for $m = 0.025$, $\delta = -1$, $V_1 = 0.04$, $V_2 = -0.04$, $\Omega = 0.3$, $\sigma = 0.01$, $l_z = 0.3$, $\kappa = 3$.

Fig. 8 depicts the variation of the IA nanopteron structure for various values of obliqueness. It is clear from the figure that the amplitude of the soliton core as well as its surrounded sinusoidal wave peaks increase with the increasing obliquity (smaller l_z). On the other hand, with the increasing of obliquity, the soliton core width and wavelength of the surrounded sinusoidal wave trends to be narrower. In addition, from the coefficients of the KdV equation we can predicted that as the IA wave approaches the direction

perpendicular to the magnetic field ($l_z = 0$), the amplitude of the soliton core suddenly becomes very high and finally the wave disappears.

5 Summary and discussions

To conclude, we have present the soliton-cnoidal wave solution of the KdV equation by using the new generalized *tanh* expansion method. Despite the simplicity of the method we used, it did provide us with some interesting results. It has been observed that the soliton core preserve its shape and velocity during the collisions with cnoidal periodic wave peaks. A surprising phenomenon was shown analytically that the phase shift of the cnoidal periodic wave is always half of its wavelength and this conclusion is universal to the soliton-cnoidal wave interaction. Based on the soliton-cnoidal wave solution, the nanopterion solution was present as a special limit case. The exact relation between the nanopterion structure and the classical soliton solution of the KdV equation was established. It was found that when the parameter m approaches zero asymptotically, the oscillating tails can be minimized and the nanopterion gets more and more closer to the classical soliton solution. In addition, the nanopterion structure was revealed in a plasma physics system which describes the propagation of IA waves. It was confirmed that the influence of plasma parameters on the nanopterion structure is in accordance with the classical soliton.

The explicit nanopterion solution obtained here may apply in many physical scenarios. For instance, the nanopterion structure can be viewed as a classical soliton with perturbations, and it may may give correction to the classical soliton in both theoretical and experimental study.

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